Effect of Initial Stresses on Surface Acoustic Waves Propagating in Infinite Elastic Plates

Ji Wang, Rongxing Wu, Jianke Du, and Dejin Huang
Piezoelectric Device Laboratory, School of Engineering, Ningbo University, 818 Fenghua Road, Ningbo, Zhejiang 315211, CHINA
E-mail: {wangji, dujianke, huangdejin}@nbu.edu.cn

***Abstract*—Initial stresses are inevitable in acoustic wave devices due to the complicated manufacturing process with ubiquitous thermal treatment procedures and environmental temperature changes. In addition, acoustic waves have been utilized as force sensors which require a good stress-frequency relationship for measurement applications. In other words, for both device performance improvement and precision sensors, a detailed analysis of the effect of initial stresses on surface acoustic wave velocity is essential in the device design and analysis. With bulk acoustic wave force sensors based on the thickness vibration mode of piezoelectric plates available as products, we are naturally encouraged to study the relationship between initial stresses and phase velocity of surface acoustic waves in a finite solid for possible applications with the advantageous higher frequency. In this study, we start from general equations of an elastic body under initial stresses for Rayleigh waves in a semi-infinite solid, and the velocity equation under initial stresses is obtained. We found that there is a good correspondence between the stress and velocity change, offering an opportunity to utilize the sensitivity of surface acoustic waves to stresses for sensor applications. We further extended the results to an elastic plate with finite thickness for the velocity and initial stress relationship in a structure close to actual surface acoustic wave resonators. We found that for plates with different thickness, the velocity versus stress exhibits a relationship similar in semi-infinite solids. Since surface acoustic wave resonators are made with piezoelectric materials such as quartz crystals, we use the ST-cut of quartz crystal to calculate the surface wave velocity versus plate thickness relations under initial stresses. These methods and procedures can be applied to other piezoelectric crystals used in acoustic wave resonators for stress sensors.**

I. INTRODUCTION

Initial stresses are inseparable with surface acoustic wave (SAW) resonators for many reasons including material processing stages and fabrication with thermal treatments inducing incompatible deformations and residual stresses and utilizing resonators as force and pressure sensors in many applications. Consequently, a precise analysis of the effect of initial stresses on the SAW resonator frequency and performance is always required for design and fabrication. Although it cannot be transparent in the consideration of initial stresses since the magnitude and distribution of initial stresses are not known, or not as simple as we have been assuming in analytical studies, solutions based on simple analytical procedures and ideal stress distributions are still useful in the evaluation and control of the process. In addition, more practical solutions can be obtained as the stress distribution is known and numerical solutions based on the actual structure with complications are obtained by following the analytical procedure. Of course, in certain applications such as force and pressure sensors, simple structure is always sought with ideal distribution of external stresses as loadings. In this case, analytical solutions will be important and applicable to actual devices.

Because the importance of initial stresses in applications, studies on the analysis of such effects have been done out with different objectives and approaches [1-7]. The general procedure is to treat the initial stresses as a finite biasing field, and then the infinitesimal vibrations of surface acoustic waves are superimposed onto the deformed elastic body under initial stresses [2, 3, 6, 7]. This has been a general approach for the analysis of acoustic wave resonators under various biasing fields such as temperature change, acceleration, and electrical field [6, 7].

In this study, we start with the formulation of elastic bodies under initial stresses by Biot [1]. For isotropic materials, relatively simple equations for SAW in both semi-infinite solids and infinite plates under initial stresses have been studied earlier [8]. The results show that there are good correspondence between external stresses and SAW velocity, thus posing an opportunity for applications in force sensors. The equations and procedures are expanded to anisotropic materials, and good correspondence between the stress amplitude and SAW velocity is obtained. The finite thickness of the plate does not change the behavior of SAW velocity under initial stress much, thus paving the way for smaller substrates in sensor applications.

II. ELASTIC SOLIDS UNDER INITIAL STRESSES

To study the effect of initial stresses on SAW in piezoelectric solids, we need to have equations and theoretical framework about the problem. Fortunately, this has been studied as a general problem in solid mechanics

1-4244-0647-1/07/$20.00 ©2007 IEEE

before with systematic research and contribution by many [1-4, 6, 7]. We shall use equations as given by Biot [1] and employed researchers [9-11] for this study without going through the derivation procedure. It is generally assumed that the initial stresses are applied first, and then the infinitesimal vibrations are superimposed onto the deformed solids.

A. Rayleigh Waves in Semi-infinite Solids Under Initial Stresses

As always, we start our study with semi-infinite solids, which have been the physical model the Rayleigh waves are originated. A typical semi-infinite solid under initial stresses is shown in Fig. 1 with material properties and wave properties.

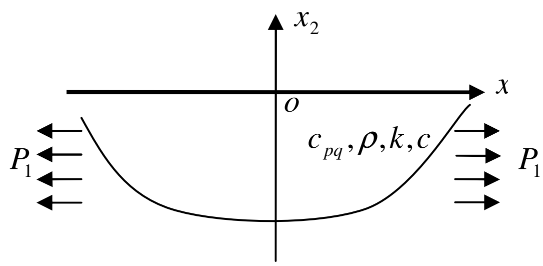

Fig. 1 A semi-infinite elastic solid under initial stresses

For the problem shown in Fig. 1, stress equations of equilibrium are

$$\sigma_{ij,j} + \left(u_{i,k}\sigma_{jk}^{(0)}\right)_{,j} = \rho\ddot{u}_i, i, j, k = 1,2,3, \tag{1}$$

where $\sigma_{ij}, u_i, \sigma_{ij}^{(0)}$, and ρ are stress tensors, displacements, initial stresses, and density of material, respectively. For a two-dimensional problem like the Rayleigh waves, (1) can be expanded to

$$\begin{aligned} &\sigma_{11,1} + \sigma_{12,2} + \left(u_{1,1}\sigma_{11}^{(0)}\right)_{,1} + \left(u_{1,2}\sigma_{22}^{(0)}\right)_{,2} = \rho\ddot{u}_1, \\ &\sigma_{21,1} + \sigma_{22,2} + \left(u_{2,1}\sigma_{11}^{(0)}\right)_{,1} + \left(u_{2,2}\sigma_{22}^{(0)}\right)_{,2} = \rho\ddot{u}_2, \\ &\sigma_{31,1} + \sigma_{32,2} + \left(u_{3,1}\sigma_{11}^{(0)}\right)_{,1} + \left(u_{3,2}\sigma_{22}^{(0)}\right)_{,2} = \rho\ddot{u}_3. \end{aligned} \tag{2}$$

If the material is isotropic, we only have two displacements coupled in the above equations; if the material is anisotropic, we shall have three coupled displacements to be considered. We emphasize this because we are going to work on anisotropic materials in this study.

We assume the displacement components are

$$u_j = A_j \exp(k\beta x_2)\exp[ik(x_1 - ct)], j = 1,2,3, \tag{3}$$

where $A_j(j = 1,2,3), k, \beta, c$, and t are the amplitudes, wavenumber, decaying index, phase velocity, and time, respectively.

With (3), we have the strain components in abbreviated notations [12-14] as

$$\begin{aligned} &S_1 = iku_1, S_2 = k\beta u_2, S_3 = 0, \\ &S_4 = k\beta u_3, S_5 = iku_3, S_6 = k\beta u_1 + iku_2. \end{aligned} \tag{4}$$

Consequently, with anisotropic materials, we have stress components in abbreviated notations as

$$\begin{aligned} &T_p = k[(c_{p6}\beta + ic_{p1})u_1 + (c_{p2}\beta + ic_{p6})u_6 + (c_{p4}\beta + ic_{p5})u_3], \\ &p = 1,2,3,4,5,6. \end{aligned} \tag{5}$$

With displacement components in (3) and stress components in (5), we can obtain the stress equations of motion through substitutions as

$$\begin{aligned} &\left(\rho c^2 + c_{66}\beta^2 + 2c_{16}i\beta - c_{11} - P_1 + \beta^2 P_2\right)A_1 + \\ &\left[c_{26}\beta^2 + (c_{12} + c_{66})i\beta - c_{16}\right]A_2 + \left[c_{46}\beta^2 + (c_{14} + c_{56})i\beta - c_{15}\right]A_3 = 0, \\ &\left[c_{26}\beta^2 + (c_{12} + c_{66})i\beta - c_{16}\right]A_1 + \\ &\left(\rho c^2 + c_{22}\beta^2 + 2c_{26}i\beta - c_{66} - P_1 + \beta^2 P_2\right)A_2 \\ &+ \left[c_{24}\beta^2 + (c_{25} + c_{46})i\beta - c_{56}\right]A_3 = 0, \\ &\left[c_{46}\beta^2 + (c_{14} + c_{56})i\beta - c_{15}\right]A_1 + \left[c_{24}\beta^2 + (c_{25} + c_{46})i\beta - c_{56}\right]A_2 \\ &\quad + \left(\rho c^2 + c_{44}\beta^2 + 2c_{45}i\beta - c_{55} - P_1 + \beta^2 P_2\right)A_3 = 0, \end{aligned} \tag{6}$$

where the initial stresses have been replaced as

$$\sigma_{11}^{(0)} = P_1, \sigma_{22}^{(0)} = P_2. \tag{7}$$

From (6), we can obtain three decaying indices, as we are familiar from the Rayleigh wave solutions, then displacements in (2) can be written as

$$\begin{aligned} &u_1 = \sum_{r=1}^{3} \alpha_{1r} C_r e^{k\beta_r x_2} e^{ik(x_1 - ct)}, \\ &u_2 = \sum_{r=1}^{3} \alpha_{2r} C_r e^{k\beta_r x_2} e^{ik(x_1 - ct)}, \\ &u_3 = \sum_{r=1}^{3} C_r e^{k\beta_r x_2} e^{ik(x_1 - ct)}, \\ &\alpha_{1r} = \frac{A_i(\beta_r)}{C_3(\beta_r)}, \alpha_{2r} = \frac{B_i(\beta_r)}{C_3(\beta_r)}, \alpha_{3r} = 1, r = 1,2,3. \end{aligned} \tag{8}$$

With (8), the stress components associated with the free surface shown in Fig. 1 in () are

$$\begin{aligned} &T_2 = k\sum_{r=1}^{3}[(c_{26}\beta_r + ic_{21})\alpha_{1r} + (c_{22}\beta_r + ic_{26})\alpha_{2r} + (c_{24}\beta_r + ic_{25})\alpha_{3r}]C_r e^{k\beta_r x_2} e^{ik(x_1 - ct)}, \\ &T_4 = k\sum_{r=1}^{3}[(c_{46}\beta_r + ic_{41})\alpha_{1r} + (c_{42}\beta_r + ic_{46})\alpha_{2r} + (c_{44}\beta_r + ic_{45})\alpha_{3r}]C_r e^{k\beta_r x_2} e^{ik(x_1 - ct)}, \\ &T_6 = k\sum_{r=1}^{3}[(c_{66}\beta_r + ic_{61})\alpha_{1r} + (c_{62}\beta_r + ic_{66})\alpha_{2r} + (c_{64}\beta_r + ic_{65})\alpha_{3r}]C_r e^{k\beta_r x_2} e^{ik(x_1 - ct)}. \end{aligned} \tag{9}$$

Now we have the displacement and stress expressions, which are needed to apply appropriate boundary conditions. Solutions like given in (9) are straightforward from Rayleigh waves and with known boundary conditions. The consideration of the initial stresses is actually hidden in (6), and cannot give an expression explicitly. As a result, the effect has to be evaluated and considered through numerical solutions to these equations.

III. NUMERICAL EXAMPLES

There are many reasons we need to consider initial stresses in SAW resonators, such as the performance change and manufacturing process evaluation, but applications in force and pressure sensors top the list in this study. For this reason, we calculated the effect of initial stresses on the phase velocity of SAW for both semi-infinite and infinite plates as substrates of anisotropic materials. These results will be useful in the design and evaluation of resonators and sensors. The consideration of infinite plates with finite thickness requires to include displacements in addition to Rayleigh waves.

A. SAW in Semi-infinite Substrates with Initial Stresses

First we consider the substrate is a semi-infinite anisotropic solid under initial stresses as shown in Fig. 1. We consider a typical semi-infinite substrate of ST-cut quartz crystal with external stress applying in the X_1 direction, which is also the direction of SAW propagation.

The boundary conditions are actually the traction-free surfaces, or we require

$$T_2(x_2=0)=T_4(x_2=0)=T_6(x_2=0)=0. \tag{10}$$

With stress components in (9), we have (10) as

$$\begin{aligned}
&\sum_{r=1}^{3}\left[(c_{26}\beta_r+ic_{21})\alpha_{1r}+(c_{22}\beta_r+ic_{26})\alpha_{2r}+(c_{24}\beta_r+ic_{25})\alpha_{3r}\right]C_r=0,\\
&\sum_{r=1}^{3}\left[(c_{46}\beta_r+ic_{41})\alpha_{1r}+(c_{42}\beta_r+ic_{46})\alpha_{2r}+(c_{44}\beta_r+ic_{45})\alpha_{3r}\right]C_r=0,\\
&\sum_{r=1}^{3}\left[(c_{66}\beta_r+ic_{61})\alpha_{1r}+(c_{62}\beta_r+ic_{66})\alpha_{2r}+(c_{64}\beta_r+ic_{65})\alpha_{3r}\right]C_r=0.
\end{aligned} \tag{11}$$

For the evaluation of (11) with initial stresses implied through decaying indices, we make the following normalization of variables

$$\overline{P}_1=P_1/c_{66},C=c/c_0,c_0=\sqrt{c_{66}/\rho}. \tag{12}$$

With substrate material as ST-cut quartz crystal, we calculated the phase velocity C versus initial stress P_1 with results shown in Fig. 2. It is clear that there are good correspondence between the magnitude of the initial stress in the wave propagation direction and the SAW phase velocity. Results like this can be used to study the phase velocity shift in resonators with initial stresses due to manufacturing and packaging, since such shifts are generally harmful to resonator performance and they should be avoided as much as possible. Of course, such results can also be used for the design and analysis of SAW sensors for force and pressure. Initial stresses in other direction may pose challenges due to the appearance of specified terms in the boundary conditions.

B. An Infinite Anisotropic Plate Under Initial Stresses

Although quartz crystal substrates in SAW resonators are relatively thick, they may not be considered as semi-infinite since we know they are finite. Consequently, it is of interests to study SAW in an infinite plate which is closer to actual a resonator. In this case, we need to consider the effect of the finite thickness as we have done before by including the exponentially growing modes in the displacements and stress terms. Consequently, the displacements are

$$\begin{aligned}
u_1&=\left(C_1\alpha_{11}e^{k\beta_1x_2}+C_2\alpha_{12}e^{-k\beta_1x_2}+C_3\alpha_{13}e^{k\beta_2x_2}+C_4\alpha_{14}e^{-k\beta_2x_2}\right)e^{ik(x_1-ct)}\\
&\quad+\left(C_5\alpha_{15}e^{k\beta_3x_2}+C_6\alpha_{16}e^{-k\beta_3x_2}\right)e^{ik(x_1-ct)},\\
u_2&=\left(C_1\alpha_{21}e^{k\beta_1x_2}+C_2\alpha_{22}e^{-k\beta_1x_2}+C_3\alpha_{23}e^{k\beta_2x_2}+C_4\alpha_{24}e^{-k\beta_2x_2}\right)e^{ik(x_1-ct)}\\
&\quad+\left(C_5\alpha_{25}e^{k\beta_3x_2}+C_6\alpha_{26}e^{-k\beta_3x_2}\right)e^{ik(x_1-ct)},\\
u_3&=\left(C_1e^{k\beta_1x_2}+C_2e^{-k\beta_1x_2}+C_3e^{k\beta_2x_2}+C_4e^{-k\beta_2x_2}+C_5e^{k\beta_3x_2}+C_6e^{-k\beta_3x_2}\right)e^{ik(x_1-ct)}.
\end{aligned} \tag{13}$$

where C_i (i=1,2,3,4) are amplitudes to be determined and the indices are from (6) in pairs. The ratios of displacement amplitudes are

$$\begin{aligned}
&\alpha_{1i}=\frac{A_i(\beta_j)}{C_i(\beta_j)},\alpha_{2i}=\frac{B_i(\beta_j)}{C_i(\beta_j)},i=2,4,6,\ j=1,2,3;\\
&\alpha_{1i}=\frac{A_i(-\beta_j)}{C_i(-\beta_j)},\alpha_{2i}=\frac{B_i(-\beta_j)}{C_i(-\beta_j)},i=2,4,6,\ j=1,2,3.
\end{aligned} \tag{14}$$

As a result, we can obtain strain components, and the stress terms can be obtained through constitutive laws.

The boundary conditions with two free surfaces are

$$\begin{aligned}
&T_2(x_2=0)=T_4(x_2=0)=T_6(x_2=0)=0,\\
&T_2(x_2=-h)=T_4(x_2=-h)=T_6(x_2=-h)=0.
\end{aligned} \tag{15}$$

It should be pointed out that the thickness of the plate is now included in equations. For simplicity, we make the following normalization of variables

$$H=h/\varsigma,k\varsigma=2\pi,kh=2\pi H, \tag{16}$$

in addition to (12). In (16), ς is the wavelength.

We now evaluate the effect of initial stress again in the wave propagation direction (X_1) with different amplitudes and thickness. We have plotted the phase velocity versus amplitudes in Figs. 3, 4, and 5.

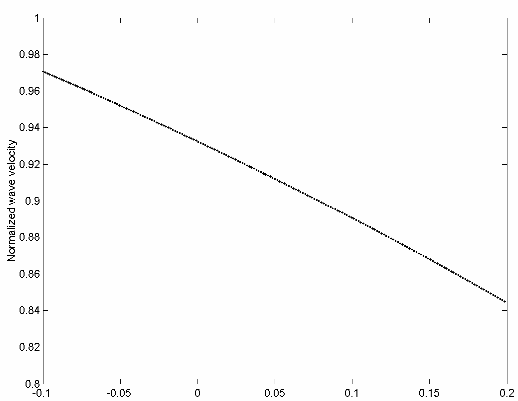

Fig. 2 Normalized SAW velocity in a semi-finite substrate of ST-cut quartz crystal under initial stress in the wave propagation direction (X_1)

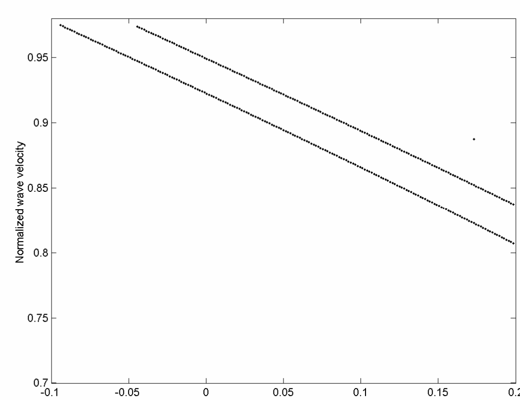

Fig. 3 Phase velocity of SAW in an ST-cut quartz crystal plate with thickness of 3 wavelengths

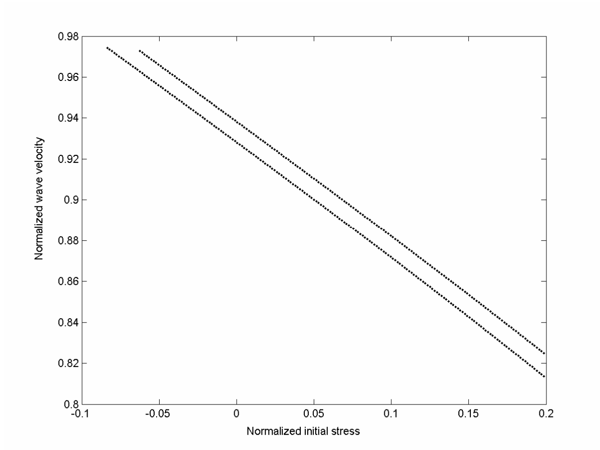

Fig. 4 Phase velocity of SAW in an ST-cut quartz crystal plate with thickness of 4 wavelengths

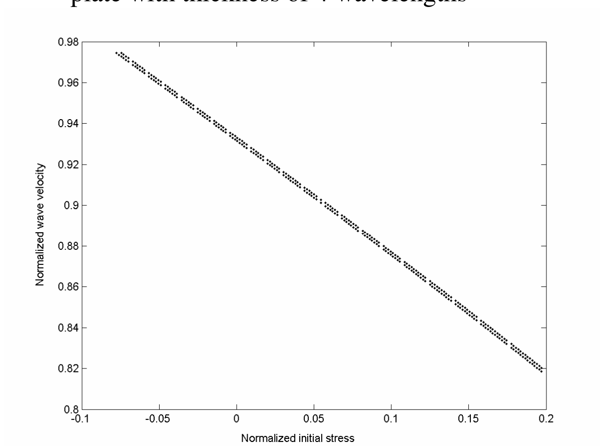

Fig. 5 Phase velocity of SAW in an ST-cut quartz crystal plate with thickness of 5 wavelengths

It is clear from above figures that if the thickness of the infinite plate is less than 5 wavelengths, we need to consider the exponentially growing modes and the SAW phase velocity will have two components. Since the two velocities will approach to Rayleigh wave velocity for thickness over 5 wavelengths, there is no advantage to use thinner substrate and the velocity of substrate over 5 wavelengths should be treated as one frequency.

IV. CONCLUSIONS

By utilizing general equations of elastic solids under initial stresses by Biot, we obtained a set of equations for SAW in anisotropic materials. The equations are further modified with the consideration of the substrates as semi-infinite solids and infinite plates with finite thickness. The main differences are in the inclusion of the exponentially growing modes in the plates and two traction-free surfaces. Consequently, there will be more equations for the determination of phase velocity. We have numerical results for anisotropic substrates of semi-infinite solids and infinite plates of ST-cut quartz crystal in terms of initial stresses and SAW velocity. The effect of initial stresses in the wave propagating direction is examined in detail. The results show that the phase velocities are closely related to amplitudes of initial stresses, exhibiting a nice correspondence suitable for sensor applications. We also found that while there is a gap between two SAW velocities for smaller plate thickness, the velocity will approach to the Rayleigh waves when the thickness is over 5 wavelengths.

Since the effect of initial stresses and other bias fields that produce stress effects, like the thermal and acceleration, are important in resonators for frequency reference and sensor applications, thorough study on the formulation and solution is of practical importance. The results presented in this paper can be improved with incremental field theory familiar in the piezoelectric acoustic wave device design and analysis.

ACKNOWLEDGMENT

This research is supported by the National Natural Science Foundation of China through Grant 10672065. Additional supports are from the Qianjiang River Fund established by Zhejiang Provincial Government and Ningbo University and administered by Ningbo University.

REFERENCES

[1] M. A. Biot, *Mechanics of Incremental Deformations*, John Wiley & Sons, Inc, New York, 1964.

[2] P.C.Y. Lee and M.S.H. Tang, "Thickness vibrations of doubly rotated crystal plates under initial deformations," *IEEE Trans. Ultra., Ferroele., and Freq. Contrl.*, **34**(6): 659-666, 1987.

[3] B.K. Sinha and H.F. Tiersten, "On the influence of uniaxial biasing stresses on the velocity of piezoelectric surface waves," *Proceedings of the 1976 IEEE Ultrasonics Symposium*, 475-480.

[4] A. Ballato, "Force-frequency comnpensation applied to four-point mounting of AT-cut resonators," *IEEE Trans. Sonics and Ultrasonics*, **SU-25**(4): 223-226, 1978.

[5] S. Ballandras and E. Bigler, "Experimental study of stress and thermoelastic effects in planar stress-compensated SAW quartz cuts," *Proceedings of the 1990 IEEE Ultrasonics Symposium*, 465-469.

[6] Jiashi Yang, *The Mechanics of Piezoelectric Structures*, World Scientific, 2006.

[7] Jiashi Yang, *Analysis of Piezoelectric Devices*, World Scientific, 2006.

[8] Ji Wang, Rongxing Wu, and Jianke Du, "Analysis of surface acosutic waves in infinite elastic plates with initial stresses," *Piezoelectricity, Acosutic Waves, and Device Applications*, World Scientific, 2007.

[9] A. M. Abd-Alla, H. A. H. Hammad, S. M. Abo-Dahab, "Rayleigh waves in a magnetoelastic half-space of orthotropic material under influence of initial stress and gravity field," *Applied Mathematics and Computation*, **154**, 583-597, 2004.

[10] S. D. Akbarov, M. Ozisik, "The influence of the third order elastic constants to the generalized Rayleigh wave dispersion in a pre-stressed stratified half-plane," *International Journal of Engineering Science*, **41**, 2047-2061, 2003.

[11] J. Su, Z.B. Kuang, H. Liu, "Love wave in ZnO/SiO2/Si structure with initial stresses," *Journal of Sound and Vibration*, **286**: 981-999, 2005.

[12] Ji Wang, Jinbo Lin, Yongping Wan, Zheng Zhong, "A two-dimensional analysis of surface acoustic waves in finite solids with the consideration of electrodes," *Inlt. J. Applied Electromagnetics and Mechanics*, **22**, 53-68, 2005.

[13] Ji Wang, Ken-ya Hashimoto, "A two-dimensional theory for the analysis of surface acoustic waves in finite elastic solids," *Journal of Sound and Vibration*, **295**, 838-855, 2006.

[14] Ji Wang, Jinbo Lin, "A two-dimensional theory for surface acoustic wave analysis in finite piezoelectric solids," *Journal of Intelligent Material Systems and Structures*, **16**(7-8), 623-629, 2005.